EE 565 Homework 2: Queueing Theory & MAC Layer

Instructor: Patrick Tague, Winter Quarter 2008

Homework Due: 10:30AM on 06 February 2008

Answers without work will receive no credit. If problems are solved using MATLAB, etc., please include code with your homework submission.

1. Let X_n be a discrete-time Markov chain such that

$$X_{n+1} = \begin{cases} X_n + 1 \bmod K & \text{with probability } p, \ (0 < p < 1), \\ X_n - 1 \bmod K & \text{with probability } 1 - p. \end{cases}$$

   - Draw the state transition diagram for the Markov chain X_n with $K = 7$.
   - Compute the transition matrix P for $K = 7$.
   - Find the stationary distribution π for $K = 7$.
   - Using MATLAB, choose a value of p and a random probability distribution π_0 (a row-vector of entries $\Pr[X_0 = x]$ for $x = 0, \ldots, K-1$) and iterate $\pi_n = \pi_{n-1}P$. Include the vectors π_n for $n = 0, 1, 2, 3, 4, 10, 20$ for $K = 7$. Repeat the simulation for various values of p and various **odd** values of K, and comment on the speed of convergence to the stationary distribution as a function of p and K. For each p and K, plot $\|\pi_n - \pi\|$ as a function of n, where $\|\cdot\|$ represents an appropriate vector norm.

2. Find the stationary distribution π for a birth-and-death queueing system with birth rate $\lambda_n = \alpha^n \lambda$ $(0 \le \alpha < 1)$ and death rate $\mu_n = \mu$ when the system is in state n. State the values of λ and μ for which the stationary distribution exists.

3. Consider a taxi stop at an airport such that taxis arrive and pick up waiting customers and customers arrive and wait for available taxis. Suppose that customers arrive with rate λ and taxis arrive with rate μ. Let $N_T(t)$ and $N_C(t)$ be the number of waiting taxis and customers at time t, respectively, and assume that either (i) $N_T(t) = N_C(t) = 0$, (ii) $N_T(t) > 0 = N_C(t)$, or (iii) $N_T(t) = 0 < N_C(t)$. In other words, customers and taxis will not be waiting simultaneously. Let $q(t) = N_C(t) - N_T(t)$, and assume that $q(0) = 0$. Show that, when taxi and customer arrivals are independent Poisson processes, the distribution of $q(t)$ is the difference between two Poisson random variables. Furthermore, use a Gaussian approximation to show that if $\lambda = \mu$ then the probability that $|q(t)| \le k$ is approximately $(2k+1)(4\pi\lambda t)^{-1/2}$ for large t.

4. Compare/contrast the performance of an $M/M/1$ queue with a single server of service rate μ to that of an $M/M/k$ queue with k servers each of service rate μ/k. Provide analytical support for your observations.

5. Compute the stationary distribution π for the discrete-time Markov chain derived in class for the number of backlogged users under the slotted ALOHA protocol. See equations (4.1)-(4.3) on page 279 of the main text.