Return to Queueing Theory

• Build on the previous material
  – From single-hop queueing delay to end-to-end queueing delay
  – Overall delay experienced in routing a packet from source to destination node

• Queueing Networks
  – Section 3.6 in text
  – Each queue represents a node which may have multiple input streams of “arrivals” from other nodes
  – Also, output stream may be split to multiple next-hop queues
  – Arrival process at one queue may depend on departure processes of multiple other queues, so probably is not Poisson
  – System is complex! What can be done?

Example

• “Transmission line” model
  – Rate $\lambda$ arrivals over one link, fixed length packets
    • Service times are constant (equal to transmission delay $1/\mu$), so $Q1$ is $M/D/1$ ($D$ means deterministic) Arrivals into $Q2$ separated by at least $1/\mu$
    • If rate of second link is $\geq$ that of first link, there is $NO$ waiting in the second queue $EVER$!
      – If packet lengths are iid exponential, this becomes $M/M/1$
  • How to characterize/analyze the second queue?
How to Evaluate Delay?

Setup:
- Assume multiple packet streams each following a unique path
- Let $x_s$ be the arrival rate (in packets/second) for stream $s$
- For circuit switching:
  - Arrivals on a link $(i,j)$ occur with rate $\lambda_{ij}$ equal to summation of $x_s$ for each stream $s$ traversing $(i,j)$
- For packet switching:
  - Include fraction $f_{ij}(s)$ of packets in stream $s$ traversing $(i,j)$

Kleinrock Independence Approximation

- 1964: Kleinrock suggests that random merger of several packet streams into a single stream restores independence of arrival times and packet lengths
- In particular, for densely connected networks with moderate-to-heavy traffic, inputs are approximately Poisson processes
Analysis under KIA

- Average number of packets in queue or service:
  - For link \((i,j)\), \(N_{ij} = \frac{\lambda_{ij}}{(\mu_{ij} - \lambda_{ij})}\) where \(1/\mu_{ij}\) is average transmit delay
  - Total packets in queue or service \(N\) is summation of \(N_{ij}\) over all links \((i,j)\)

- Average queueing delay:
  - From Little’s Theorem, \(T = \frac{N}{\gamma}\) where \(\gamma\) is sum rate of all \(x_s\) arrival streams

- Including proc/prop delay \(d_{ij}\) at each \((i,j)\):
  - Add the term \(\lambda_{ij}d_{ij}\) in the summation

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Time Reversibility

- Let \(X\) be a DTMC, and let \(X_n\) denote the state at time \(n > 0\). Let’s look at the behavior backward in time, instead of looking forward in time.

- Let \(P^*_{ij}\) be the reverse transition probability given by \(\Pr[X_m = j \mid X_{m+1} = i]\).

- Show that \(\pi_i P^*_{ij} = \pi_j P_{ji}\)

- Def: A DTMC \(X\) is **time reversible** if \(P^*_{ij} = P_{ji}\). In this case, the stationary distribution of the reversed DTMC \(X^*\) is the same as the forward chain \(X\).
  - We immediately see that a DTMC \(X\) is time reversible if and only if the detailed balance equations hold.

- This extends in a straightforward way to CTMCs.
Burke’s Theorem

- **Theorem (Burke):** Consider an $M/M/1$, $M/M/k$, or $M/M/\infty$ queue with arrival rate $\lambda$, and suppose the system starts in steady-state. Then:
  - The departure process is Poisson with rate $\lambda$.
  - The number of customers in the system at time $t$ is independent of the sequence of departure times prior to $t$.

- Why is this true?
  - $M/M/1$, $M/M/k$, and $M/M/\infty$ are time reversible CTMCs.
  - Departures prior to time $t$ are arrivals after time $t$ in reversed process, which is Poisson, so future arrivals do not depend on system occupancy.

Example – Revisited

- Returning to the previous example:
  - Suppose $Q1$ is $M/M/1$ with exp. service rate $\mu_1$.
  - From Burke’s Theorem, arrivals to $Q2$ are Poisson with rate $\lambda$ and exp. service rate $\mu_2 \rightarrow M/M/1$.
  - $\pi_n^{(i)} = \rho_i^n(1-\rho_i)$, $\rho_i = \lambda/\mu_i$, $i=1,2$
  - From 2nd part of Burke’s Theorem, number of customers in each queue is independent, so $\Pr[n \text{ in } Q1, m \text{ in } Q2] = \pi_n^{(1)} \pi_m^{(2)}$
  - This is a simple example of an acyclic queueing network
Queueing Networks

• Setup:
  – A network of $K$ queues, each with a single server, service rate $\mu_i$ at queue $i$
  – Arrivals into the network at queue $i$ follow a Poisson process with rate $r_i$. At least one $r_i$ is positive.
  – Customer served by queue $i$ moves to queue $j$ with probability $P_{ij}$ – leaves with residual probability
  – Total arrival rate into queue $j$ is
  
  \[ \lambda_j = r_j + \sum_{i=1}^{K} \lambda_i P_{ij}, \quad \text{for } j = 1, \ldots, K \]
  – Let $\rho_j = \lambda_j / \mu_j$

Queueing Network State Space

• Let $n = (n_1, \ldots, n_K)$ denote a state in the state space $\mathcal{X}^K$, where $\mathcal{X}$ is the state space of each queue (here, the non-negative integers), i.e. $n_i =$ number of customers in queue $i$.

• State transitions (with very high probability):
  – New arrival at queue $j$ (with rate $r_j$)
    • State $n(j^+) = n + e_j = (n_1, \ldots, n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_K)$
  – Exit system from queue $j$ (with rate $\rho_j \left( 1 - \sum_{i=1}^{K} r_i \right)$)
    • State $n(j^-) = n - e_j = (n_1, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_K)$
  – Customer moves from $j$ to $i$ (with rate $\mu_j P_{ji}$)
    • State $n(j^-, i^+) = n - e_j + e_i$
Jackson’s Theorem

- **Theorem (Jackson):** For such a queueing network, if $\rho_j < 1$ for $j=1,...,K$, then for all $n=(n_1,...,n_K), n_j \geq 0$, we have

$$\pi_n = \prod_{j=1}^{K} \pi_{n_j}^{(j)}$$

$$\pi_{n_j}^{(j)} = \rho_j^{n_j} (1 - \rho_j)$$

- In other words, queueing network behaves as independent collection of $M/M/1$ queues even though the arrival to each queue is not Poisson.

Closed Queueing Networks

- Same setup as before, only $r_i=0$ and $P_{ij}$ sum to 1 over $j$ for all $i=1,...,K$, i.e. nobody enters and nobody leaves, so

$$\lambda_j = \sum_{i=1}^{K} \lambda_i \rho_{ij}, \quad \text{for } j = 1, \ldots, K$$

and total number of customers is fixed at $M$.

- Here, $\pi_n$ can be positive only if

$$n_1 + \ldots + n_K = M.$$